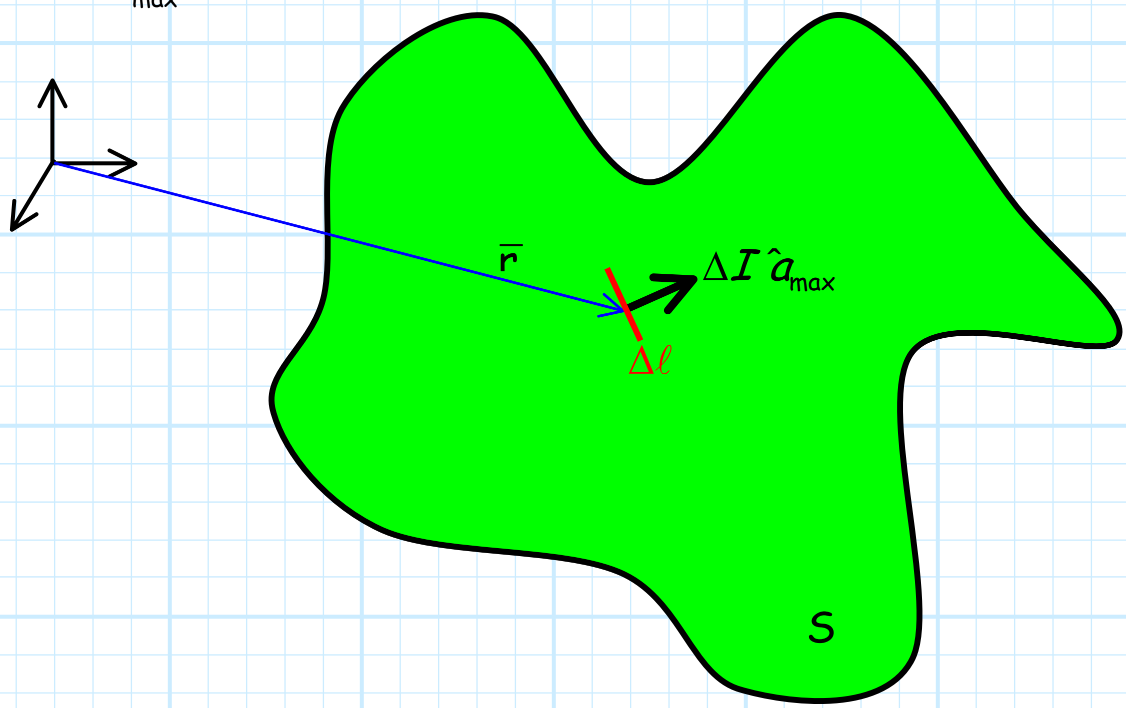
Surface Current Density

Consider now the problem where we have moving **surface** charge $\rho_s(\bar{r})$.

The result is **surface** current!

Say at a given point $\bar{r}$ located on a surface S, charge is moving in **direction** $\hat{a}_{max}$.



Now, consider a **small length** of contour $\Delta\ell$ that is centered at point $\bar{r}$, and oriented such that it is orthogonal to unit vector $\hat{a}_{max}$. Since charge is moving across this small length, we can define a **current** ΔI that represents the current flowing across $\Delta\ell$.

Note **vector** $\Delta I\ \hat{a}_{max}$ therefore represents both the **magnitude** (ΔI) and **direction** $\hat{a}_{max}$ of the current flowing across contour $\Delta\ell$ at point $\bar{r}$.

From this, we can define a **surface current density** $\mathbf{J}_s(\bar{r})$ at every point $\bar{r}$ on surface S by **normalizing** $\Delta I\ \hat{a}_{max}$ by dividing by the length $\Delta\ell$:

$$\mathbf{J}_s(\bar{r}) = \lim_{\Delta\ell \to 0} \frac{\Delta I\ \hat{a}_{max}}{\Delta\ell} \qquad \left[\frac{\text{Amps}}{\text{m}}\right]$$

The result is a **vector** field !

NOTE: *The unit of **surface** current density is current/**length**; for example, A/m.*

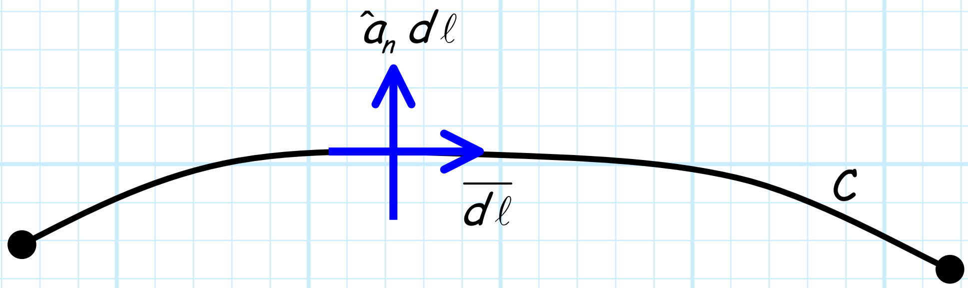
Given that we know surface current density $\mathbf{J}_s(\bar{r})$ throughout some volume, we can find the total **current** across **any** arbitrary **contour *C*** as:

$$I = \int_C \mathbf{J}_s(\bar{r}) \cdot \hat{a}_n d\ell$$

This looks very much like the contour integral we studied in the previous chapter. However, there is one **big** difference!

The differential vector $\hat{a}_n \, d\ell$ is a vector that tangential to **surface** S (i.e., it lies on surface S), but is **normal** to contour C!

This of course is the **opposite** of the differential vector $\overline{d\ell}$ in that $\overline{d\ell}$ lies **tangential** to the contour:



As a result, we find that $\overline{d\ell} \cdot \hat{a}_n \, d\ell = 0$. However, note the **magnitude** of each vector is identical:

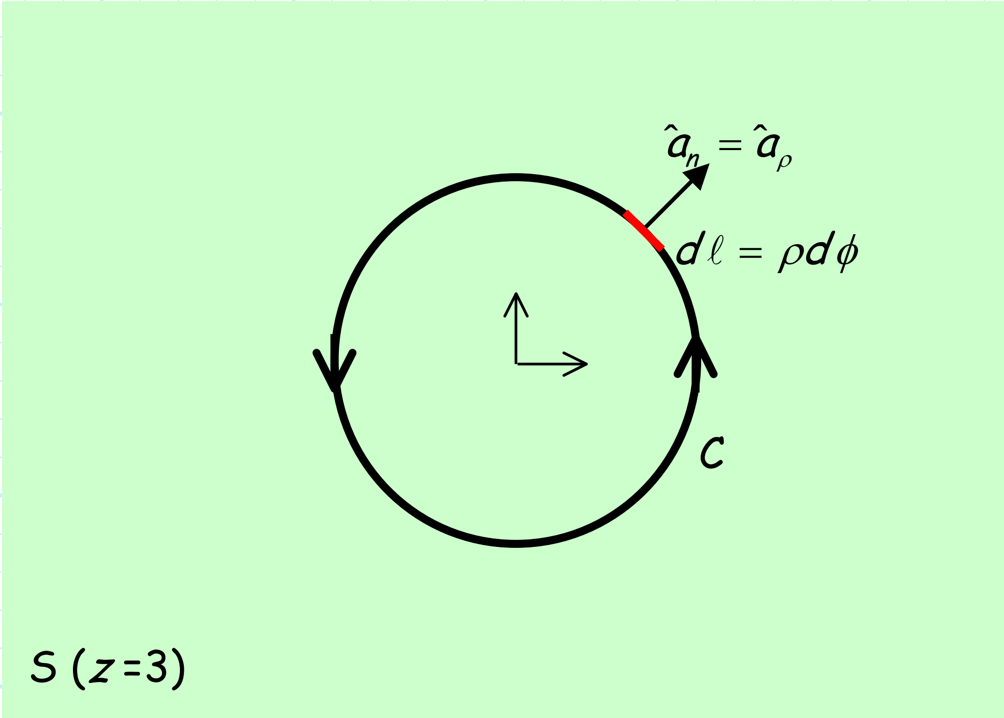
$$\left|\overline{d\ell}\right| = \left|\hat{a}_n \, d\ell\right| = d\ell$$

For example, consider the planar surface z=3. On this surface is a contour that is a **circle**, radius 2, centered around the z-axis.

For the contour integrals we studied in Section 2-5, we would use:

$$\overline{d\ell} = \hat{a}_\phi \, \rho \, d\phi$$

However, to determine the total current flowing across the contour, we use $\hat{a}_n = \hat{a}_\rho$ and $d\ell = \rho \, d\phi$. Note the **directions** of these two differential vectors are **different**, but their **magnitudes** are the **same**.



The integral for determining the **total** current flowing from **inside** the circle to **outside** the circle is therefore:

$$
\begin{aligned}
I &= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_n \, d\ell \\
&= \int_C \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho \, \rho \, d\phi \\
&= \int_0^{2\pi} \mathbf{J}(\bar{\mathrm{r}}) \cdot \hat{a}_\rho 2 \, d\phi
\end{aligned}
$$